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FURTHER DEVELOPMENT OF A STRATEGIC WEAPONS EXCHANGE ALLOCATION MODEL

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January 1978

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This paper describes a model of a two-strike nuclear exchange that requires the solution of a nonlinear maxmin problem. The model is approximated by a separable, piecewise linear model for which the resulting maxmin problem can be solved using a branch and bound algorithm. Computational results are discussed.

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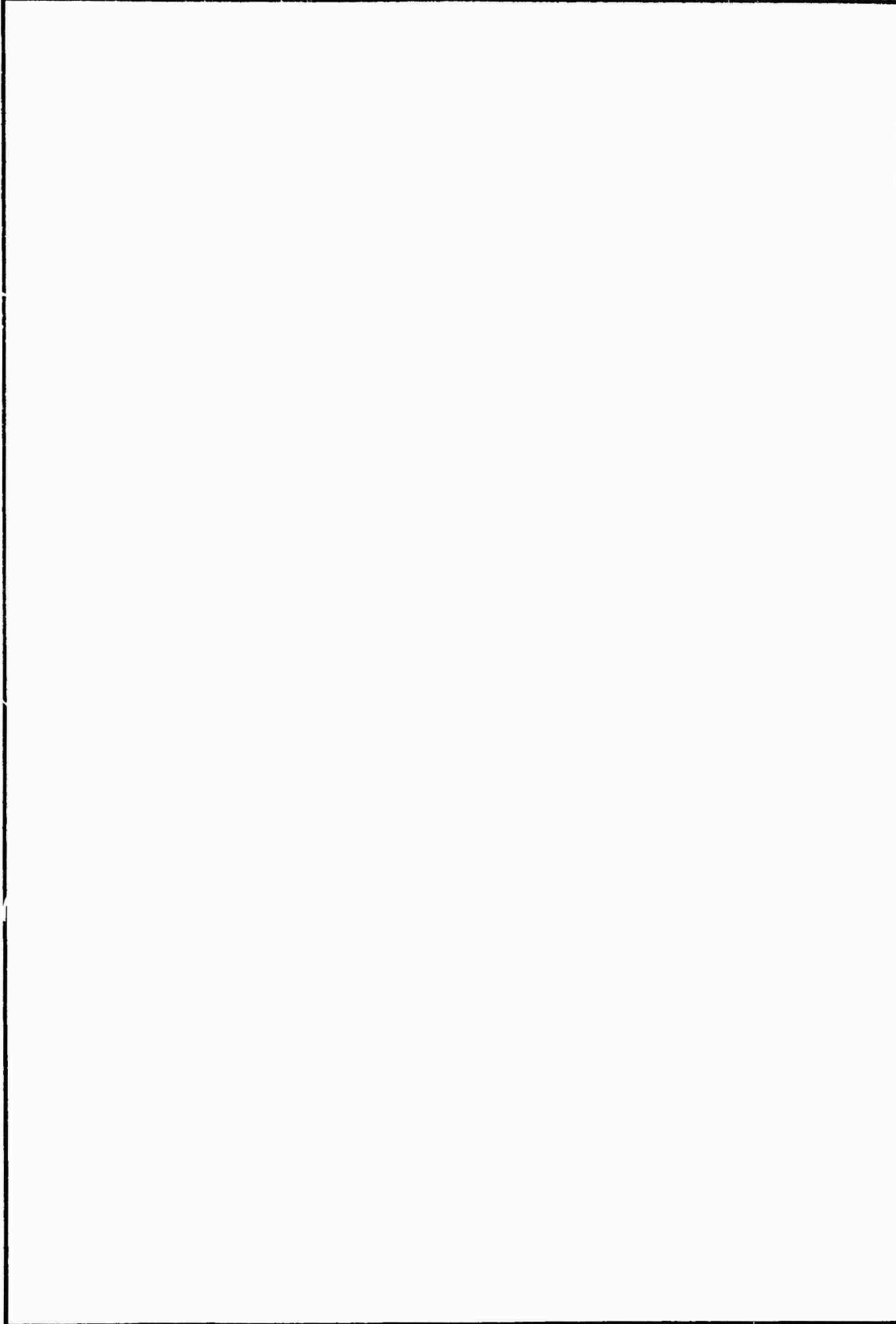
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FOREWORD

This paper examines a two-sided strategic nuclear exchange model involving ICBMs, SLBMs, bombers and societal value on each side. The first striker allocates his weapons against the second striker's weapons and value in such a way that, after massive retaliation by the second striker, the difference between the second striker's value destroyed and that of the first striker is maximized. The paper presents computational results including the sensitivities of the outcomes to various force characteristics.

The paper is a contribution to the literature in that it incorporates realistic weapon-on-weapon attrition functions into a formulation of the two-strike strategic exchange problem for which globally optimal allocations can be guaranteed.

Chapter I

INTRODUCTION

Strategic weapons planning models based on two-strike nu-
clear exchanges have been an integral part of strategic analysis
since the early 1960s. One line of research in this area has
been the development of models for which one could find "optimal"
allocations of weapons to targets. These models frequently took
the form of max-min problems wherein the first striker has a
choice of allocating his weapons against his opponent's value
targets (economic and industrial targets), thereby obtaining
immediate benefit, or of allocating those weapons against his
opponent's weapons, in the hope of reducing the impact of the
expected retaliation. This question of balancing counterforce
and countervalue targeting was perhaps academic when strategic
weapons were relatively invulnerable. However, with improved
ICBM accuracy, this issue becomes increasingly significant.

Among the approaches used in the past to answer this
problem were the Strategic Weapons Exchange Models, reported
in [1] and the Arsenal Exchange Model [2]. Neither of these
models could guarantee the globally optimal solution of the
max-min problem they considered. Both incorporated Lagrangian
solution procedures.

A branch-and-bound procedure was described by Bracken,
Falk, and Miercort in [3] in which they constructed a piecewise-
linear approximation to their original two-strike nuclear ex-
change model and found the global solution to the piecewise-
linear model using an algorithm of Falk [5]. The Bracken, Falk,

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Miercort paper established that, *in principle*, this approach could be practically applied in two-strike modelling.

This paper extends the Bracken, Falk, Miercort approach to a model of more interesting detail, explicitly modelling ICBMs, submarines, and bombers. Realistic attrition equations are developed, and the results of numerous computer runs of the model are presented to demonstrate some of the remarkable responses of the model to parameter excursions.

Chapter II

THEORETICAL DEVELOPMENT

Let us call the first striker Red and his opponent Blue.
Our scenario goes as follows: Red strikes first with all his
weapons on Blue's weapons and value, where the term "value"
represents Blue's capability to sustain its society, and may
include population, industry, and so forth. Blue then retaliates
with all his remaining weapons against Red value. This inter-
action is depicted in Figure 1.
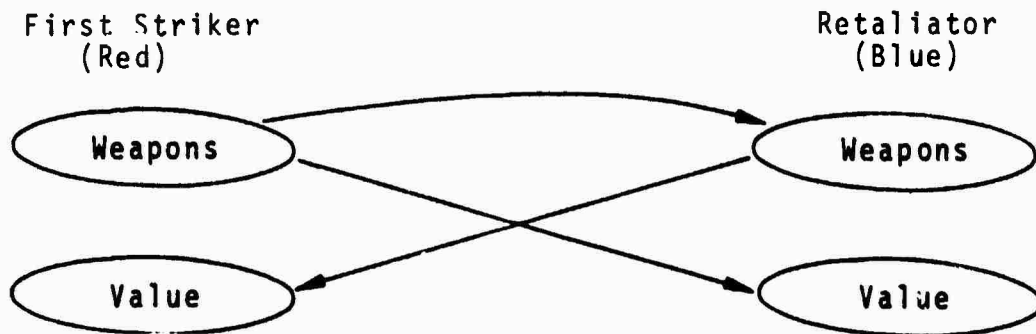
Figure 1. STRATEGIC EXCHANGE

Red has two concerns: first, to destroy Blue's value and
second, to preserve his own value through destruction of Blue's
retaliatory weapons. To investigate this further, we develop
some notation.

We will denote Red's allocation by $\underline{x} = \{x_{ij}: i=1,\ldots,I;$
$j=1,\ldots,J\}$, where x_{ij} is the number of Red's class-i weapons
directed against Blue class-j weapons for $j=1,\ldots,J-1$, and x_{iJ}
is the number of Red's class-i weapons directed against Blue
value. Let $B_j(\underline{x})$ represent the number of Blue's class-j weapons

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that survive an attack $\underline{x}$, for $j=1,\ldots,J-1$, and let $\underline{y} = \{y_1,\ldots,y_{J-1}\}$ be some Blue retaliation. If $D_B(\underline{x})$ represents the damage to Blue value resulting from attack $\underline{x}$ and $D_R(\underline{y})$ is the damage to Red value resulting from Blue retaliation $\underline{y}$, then we use as our objective function the difference

$$D_B(\underline{x}) - D_R(\underline{y}).$$

Blue desires to minimize this difference for any given $\underline{x}$, while Red wishes to choose his attack $\underline{x}$ so that this expression is maximized, taking Blue's retaliation into account. If m_1, $i=1,\ldots,I$ represents Red's initial inventory of class-i weapons, then this two-strike war can be modelled by the following mathematical programming problem

$$\max_{\underline{x}} \min_{\underline{y}} \quad D_B(\underline{x}) - D_R(\underline{y})$$

subject to

$$\sum_{j=1}^{J} x_{ij} \le m_i \qquad i=1,\ldots,I$$

$$y_j \le B_j(\underline{x}) \qquad j=1,\ldots,J-1$$

$$x_{ij}, y_j \ge 0 \qquad i=1,\ldots,I; \; j=1,\ldots,J.$$

This problem can be greatly simplified if we make the rather weak restriction that $D_R(\underline{y})$ be monotone nondecreasing in each component of y; i.e., if Blue retaliates with any more of any weapon type, then the damage to Red's value does not decrease. With this restriction the problem becomes

$$\max \; D_B(\underline{x}) - D_R\left(\underline{B}(\underline{x})\right)$$

subject to

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$$\sum_{j=1}^{J} x_{ij} \leq m_i \qquad i=1,\ldots,I$$

$$x_{ij} \geq 0 \qquad i=1,\ldots,I; \; j=1,\ldots,J$$

$$B(\underline{x}) = \{B_1(\underline{x}), \; B_2(\underline{x}), \ldots, B_{J-1}(\underline{x})\}.$$

This reformulation is crucial to our approach. The next two sections develop the functions $B_1(\underline{x}),\ldots,B_{J-1}(\underline{x})$, $D_R(\underline{y})$, and $D_B(\underline{x})$.

Chapter III

WEAPON-ON-WEAPON INTERACTIONS

This chapter develops formulae for attrition of Blue weapon resources resulting from a Red attack.

For the purposes of this paper, we will allow Red and Blue four weapon classes each.[1] The weapon classes, with their distinguishing characteristics, are as follows:

> **Class 1--ICBMs.** ICBMs are normally housed in silos, which are attacked individually. The locations of the silos are assumed to be known, and they may or may not be defended.
>
> **Class 2--SLBMs-AT-SEA.** SLBMs-at-sea comprise those missiles on submarines that are patrolling on station. Our model permits these submarines to be attacked by strategic weapons. Submarines are attacked individually, the attacker is uncertain of their location, and they are not defended.
>
> **Class 3--BOMBERS.** Bombers are attacked in groups, at airfields of known location, where they may or may not be defended. (A "group" of weapons is a configuration such that an attack against any one of them is an attack against all of them.) Unlike missiles, bombers may be scrambled on warning and called back, if necessary. The model assumes five or fewer separate Blue airfields.
>
> **Class 4--SLBMs-IN-PENS.** This class consists of SLBMs in submarines that are available for strategic retaliation but are in pens of known location. Thus, they must put to sea before they can retaliate effectively. All submarines at any given pen are attacked as a group and may or may not be defended. We allow for three or fewer Blue pens.

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[1]Problems involving more weapon classes can be and have been solved. For instance, the IDASNEM model [4] developed at the Institute for Defense Analyses, using the approach outlined in this paper, allows nine weapon classes per side.

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If Red attacks Blue, and there are four weapon classes on each side, then there is a maximum of sixteen weapon-on-weapon interactions. However, because one would pay a price in prolonged run times if one were to allow all interactions, it is desirable to disallow, *a priori*, certain unlikely weapon-on-weapon interactions. For instance, it is unlikely that Red bombers would be used against Blue ICBMs, since there would be sufficient warning before the bombers' arrival to launch all the ICBMs, thereby removing them as a target. All weapons may, of course, attack value targets.

We will assume that Red ICBMs and SLBMs-at-sea may attack any Blue weapon class. Red bombers, however, will only be effective against Blue SLBMs-in-pens, provided we assume that the latter require sufficient time to put to sea and so are vulnerable to bombers. Red SLBMs-in-pens may only attack value targets. These relationships are summarized in Figure 2.
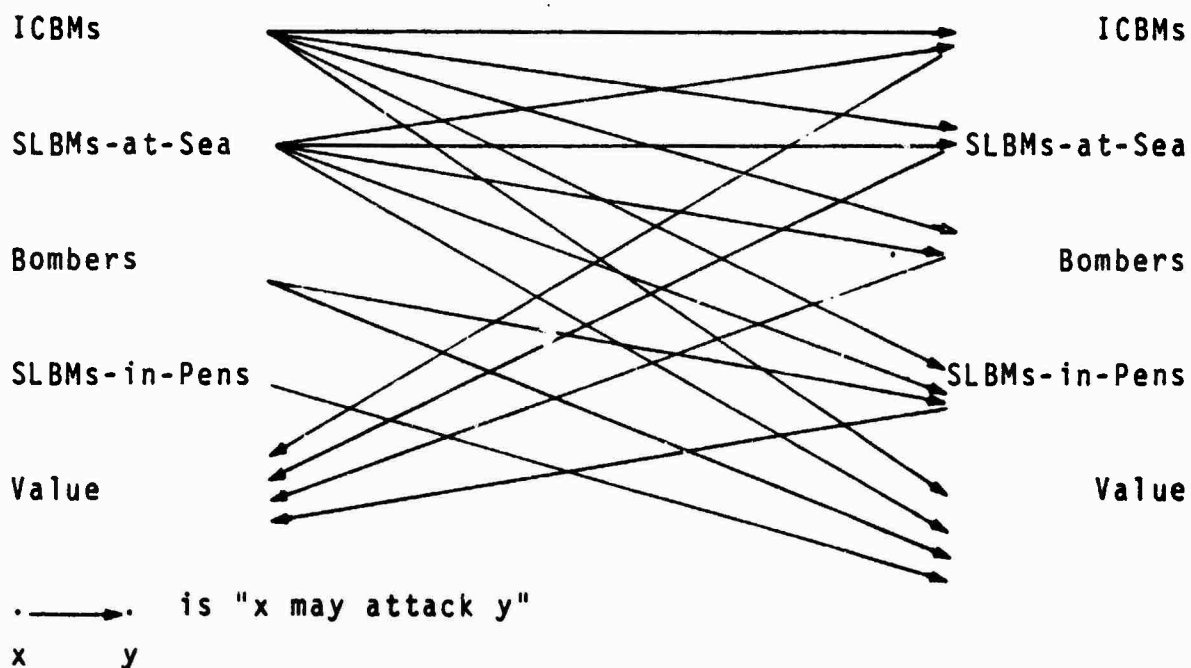
Figure 2. WEAPON-ON-WEAPON INTERACTIONS

The following set of equations gives the Blue surviving force after a Red attack. The attacking units mentioned below are assumed to be warheads--when the attacking weapon is a missile--or bombs (or SRAMS, for appropriate parameter values) when the attacking weapon is a bomber. Note that we are replacing what are essentially random variables with their expectations.

A. SURVIVING BLUE ICBM WARHEADS

n^I: number of silos,

p_r^I: probability of single shot kill of a silo due to a single attacking unit of class r,

ρ_r: individual reliability of an attacking unit of class r,

q_r: overall reliability of weapon class r; i.e., there is a probability $1-q_r$ that no weapons of class r will function,

σ_r^I: probability that an attacking unit of class r will penetrate silo defenses,

w^I: number of warheads per missile,

x_{r1}:   number of attacking units of class r assigned to category "ICBMs."

We require that attacking units of any single type be uniformly distributed among the targets of a single category. Therefore, the number of attacking units assigned to each silo is x_{r1}/n^I. Then, the probability that an individual silo survives an attack by weapon class r, provided weapon class r does not fail, is

$$\left(1-\rho_r \; \sigma_r^I \; p_r^I\right)^{x_{r1}/n^I} \; .$$

Taking into account the probability that weapon class r might fail, the probability that a silo survives an attack by weapon class r is

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$$(1-q_r) + q_r\left(1-\rho_r \ \sigma_r^I \ p_r^I\right)^{x_{r1}/n^I} .$$

Therefore, recalling that only Red weapon classes 1 and 2 will attack Blue ICBMs, the probability of a silo surviving the full attack is

$$\prod_{r=1}^{2} \left\{(1-q_r) + q_r\left(1-\rho_r \ \sigma_r^I \ p_r^I\right)^{x_{r1}/n^I}\right\}$$

and so the approximate expected number of surviving Blue ICBM warheads is

$$w^I n^I \prod_{r=1}^{2} \left\{(1-q_r) + q_r\left(1-\rho_r \ \sigma_r^I \ p_r^I\right)^{x_{r1}/n^I}\right\} .$$

B. SURVIVING BLUE SLBM-AT-SEA WARHEADS

n^S: number of submarines-at-sea

τ^S: probability that Red will correctly acquire the location of an individual submarine,

p_r^S: probability of single shot kill of a submarine due to a single attacking unit of class r,

ρ_r: individual reliability of an attacking unit of class r,

q_r: overall reliability of class r,

w^S: number of warheads per missile,

γ^S: number of missiles per submarine,

x_{r2}: number of attacking units of class r assigned to the category "SLBMs-at-sea."

As with ICBMs, we assume that the number of class-r attacking units assigned to each submarine is x_{r2}/n^S. Considering only class-r attacking units, the probability that a single submarine will survive is

$$\left(1-\rho_r \; p_r^S\right)^{x_{r2}/n^S} \quad ,$$

if weapon class r does not fail. Taking this possibility into account, the probability of submarine survival is

$$(1-q_r) \; + \; q_r \left(1-\rho_r \; p_r^S\right)^{x_{r2}/n^S} \quad .$$

Now, the probability that the submarine will survive an attack by Red weapon classes 1 and 2 is

$$\prod_{r=1}^{2} \left\{ (1-q_r) \; + \; q_r\left(1-\rho_r \; p_r^S\right)^{x_{r2}/n^S} \right\} \quad .$$

All the above assumes that the location of the submarine is known. Considering the possibility that the submarine's location will not be acquired, the probability of submarine survival is

$$(1-\tau^S) \; + \; \tau^S \prod_{r=1}^{2} \left\{ (1-q_r) \; + \; q_r\left(1-\rho_r \; p_r^S\right)^{x_{r2}/n^S} \right\} \quad .$$

Hence the approximate number of surviving Blue SLBM-at-sea warheads is

$$n^S w^S \gamma^S \left\{ (1-\tau^S) \; + \; \tau^S \prod_{r=1}^{2} \left\{ (1-q_r) \; + \; q_r\left(1-\rho_r \; p_r^S\right)^{x_{r2}/n^S} \right\} \right\} \quad .$$

C. SURVIVING BLUE BOMBS

$\pi_n^{r_3}$: number of bombers at airfield n (n=1,...,5) (this may be 0 when there are fewer than 5 airfields),

η_n^B: probability that a bomber will successfully scramble from airfield n,

$p_{n,r}^B$: probability that a bomber at airfield n will be destroyed when airfield n is attacked by a single class-r attacking unit,

ρ_r: individual reliability of a class-r attacking unit,

q_r: overall reliability of weapon class r,

$\sigma_{n,r}^B$: probability that a class-r attacking unit will penetrate the defenses of airfield n,

γ_n^B: number of bombs per bomber at airfield n,

x_{r3}: number of class-r weapons allocated to "bombers" ($r = 1, 2$).

We will require that attacking units be allocated proportionally to the number of bombers at each airfield, thus airfield n receives

$$m_n^B \; x_{r3} \Big/ \sum_{n=1}^{5} m_n^B$$

weapons of class r. The probability that a bomber at airfield n survives a class-r attack, assuming no scramble and no overall failure of weapon class r, therefore is

$$\left(1 - \rho_r \; \sigma_{n,r}^B \; p_{n,r}^B\right)^{m_n^B \; x_{r3} \big/ \sum_{n=1}^{5} m_n^B} .$$

Including the probability of successful scramble, the probability of survival is

$$\eta_n^B + (1 - \eta_n^B) \left(1 - \rho_r \; \sigma_{n,r}^B \; p_{n,r}^B\right)^{m_n^B \; x_{r3} \big/ \sum_{n=1}^{5} m_n^B} .$$

Recalling that the probability that weapon class r will fail as a class is $1 - q_r$, the probability of survival of a single bomber is

$$(1 - q_r) + q_r \left\{ \eta_n^B + (1 - \eta_n^B) \left(1 - \rho_r \; \sigma_{n,r}^B \; p_{n,r}^B\right)^{m_n^B \; x_{r3} \big/ \sum_{n=1}^{5} m_n^B} \right\} .$$

Hence, the probability that a single bomber will survive the full attack is

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$$\prod_{r=1}^{2}\left\{(1-q_r) + q_r\left\{\eta_n^B + (1-\eta_n^B)\left(1-\rho_r\ \sigma_{n,r}^B\ p_{n,r}^B\right)^{m_n^B\ x_{r3}/\sum_{n=1}^{5} m_n^B}\right\}\right\}.$$

The approximate number of bombs surviving at airbase n is $\gamma_n^B m_n^B$ times the above expression, so that the total number of bombs remaining is

$$\sum_{n=1}^{5}\gamma_n^B m_n^B \prod_{r=1}^{2}\left\{(1-q_r) + q_r\left\{\eta_n^B + (1-\eta_n^B)\left(1-\rho_r\ \sigma_{n,r}^B\ p_{n,r}^B\right)^{m_n^B x_{r3}/\sum_{n=1}^{5} m_n^B}\right\}\right\}.$$

D.   SURVIVING BLUE SLBM-IN-PENS WARHEADS

m_n^P: number of submarines in pen n (n=1,2,3) (may be 0 when there are fewer than 3 pens),

$p_{n,r}^P$: probability of single shot kill of a submarine in pen n due to a class-r attacking unit,

ρ_r: individual reliability of a class-r attacking unit,

q_r: overall reliability of class r,

$\sigma_{n,r}^P$: probability that a class-r attacking unit will penetrate the defenses of pen n,

w_n^P: number of warheads per missile in pen n,

γ_n^P: number of missiles per submarine in pen n,

x_{r4}: number of class-r attacking units assigned to "SLBMs-in-pens" (r=1,2,3).

As with bombers, pens will be attacked in proportion to the number of submarines in each. Thus, the number of class-r attacking units assigned to pen n is

$$m_n^P\ x_{r4}/\sum_{n=1}^{3} m_n^P\ .$$

If class r does not fail, the probability that a submarine in pen n will survive all class-r attacking units is

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$$\left(1 - \rho_r \; \sigma_{n,r}^P \; p_{n,r}^P \right)^{m_n^P \; x_{r4} / \sum\limits_{n=1}^{3} m_n^P} .$$

Including the overall reliability, the probability that a submarine will survive is

$$\left\{ (1-q_r) + q_r \left(1 - \rho_r \; \sigma_{n,r}^P \; p_{n,r}^P \right)^{m_n^P \; x_{r4} / \sum\limits_{n=1}^{3} m_n^P} \right\} .$$

Thus, the probability of the submarine surviving the full attack is

$$\prod_{r=1}^{3} \left\{ (1-q_r) + q_r \left(1 - \rho_r \; \sigma_{n,r}^P \; p_{n,r}^P \right)^{m_n^P \; x_{r4} / \sum\limits_{n=1}^{3} m_n^P} \right\} ,$$

so that the approximate number of SLBM-in-pens warheads that survive is

$$\sum_{n=1}^{3} w_n^P \gamma_n^P m_n^P \prod_{r=1}^{3} \left\{ (1-q_r) + q_r \left(1 - \rho_r \; \sigma_{n,r}^P \; p_{n,r}^P \right)^{m_n^P \; x_{r4} / \sum\limits_{n=1}^{3} m_n^P} \right\} .$$

Chapter IV

VALUE DAMAGE FUNCTIONS

For the purposes of this paper, let us suppose that damage to value on both sides can be adequately described by the following two functions:

$$D_B(\underline{x}) = V_B\left(1-\exp\left(-\sum_{i=1}^{4} \mu_i^B \, x_{15}^{\nu_i^B}\right)\right),$$

$$D_R(\underline{y}) = V_R\left(1-\exp\left(-\sum_{j=1}^{4} \mu_j^R \, y_j^{\nu_j^R}\right)\right),$$

where V_B and V_R are the aggregate Blue and Red values and the parameters μ_i^B, μ_j^R, ν_i^B and ν_j^R have been determined. This determination can be made using curve fitting techniques on the results of a one-sided allocation model for samples of possible Red attacks and Blue retaliations. It is not even a requirement of this approach that these functions be expressible in explicit form. For instance, the IDASNEM model uses the output of a subroutine that assesses the damage against up to two thousand target classes per side.

Chapter V
SOLVING THE MODEL

The foregoing allows us to write explicitly the model:

$$\underset{\underline{x}}{\text{maximize}} \quad V_B\left(1-\exp\left(-\sum_{i=1}^{4} \mu_i^B \, x_{i5}^{\ \nu_i^B}\right)\right) - V_R\left(1-\exp\left(-\sum_{j=1}^{4} \mu_j^R \, y_j^{\ \nu_j^R}\right)\right)$$

subject to

$$\sum_{j=1}^{5} x_{ij} \le m_i \qquad i=1,\ldots,4$$

$$y_1 = w^I n^I \prod_{i=1}^{2} \left\{(1-q_i) + q_i\left(1-\rho_i \ \sigma_i^I \ p_i^I\right)^{x_{i1}/n^I}\right\}$$

$$y_2 = n^S w^S \gamma^S \left\{(1-\tau^S) + \tau^S \prod_{i=1}^{2} \left\{(1-q_i) + q_i\left(1-\rho_i \ p_i^S\right)^{x_{i2}/n^S}\right\}\right\}$$

$$y_3 = \sum_{n=1}^{5} \gamma_n^B m_n^B \prod_{i=1}^{2} \left\{(1-q_i) + q_i\left\{\eta_n^B + (1-\eta_n^B)\left(1-\rho_i \ \sigma_{n,i}^B \ p_{n,i}^B\right)^{m_n^B \, x_{i3}/\sum_{k=1}^{5} m_k^B}\right\}\right\}$$

$$y_4 = \sum_{n=1}^{3} w_n^P \gamma_n^P m_n^P \prod_{i=1}^{3} \left\{(1-q_i) + q_i\left(1-\rho_i \ \sigma_{n,i}^P \ p_{n,i}^P\right)^{m_n^P \, x_{i4}/\sum_{k=1}^{3} m_k^P}\right\}$$

$$x_{ij} \ge 0 \quad i=1,\ldots,4 \quad j=1,\ldots,5$$

$$y_j \ge 0 \quad j=1,\ldots,4$$

We solve this problem using the approach of [3]; that is, we first introduce new variables and perform some elementary

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manipulations to put the problem in separable form--in which each function $g(\underline{\zeta})$, where $\underline{\zeta} = (\zeta, \ldots, \zeta_m)$, is of the form

$$g(\underline{\zeta}) = \sum_{i=1}^{m} g_i(\zeta_i).$$

In separable form, this problem is

$$\underset{\underline{x} \ \underline{z} \ \xi^R \xi^B \underline{C}^B \underline{C}^P}{\text{maximize}} \quad \left\{ V_B\left(1-\exp(-\xi^B)\right) - V_R\left(1-\exp(-\xi^R)\right) \right\}$$

subject to

$$\xi^B \leq \sum_{i=1}^{4} \mu_i^B \ x_{i5} \ ^{\nu_i^B}$$

$$\xi^R \geq \sum_{j=1}^{4} \mu_j^R \ z_j \ ^{\nu_j^R}$$

$$\sum_{j=1}^{5} x_{ij} \leq m_i \qquad i=1,\ldots,4$$

$$\ln z_1 - \ln(w^I n^I) \geq \sum_{i=1}^{2} \ln\left\{(1-q_i) + q_i\left(1-\rho_i \ \sigma_i^I \ p_i^I\right)^{x_{i1}/n^I}\right\}$$

$$\ln\left[\left(\frac{z_2}{n^S w^S \gamma^S} - (1-\tau^S)\right)\frac{1}{\tau^S}\right] \geq \sum_{i=1}^{2} \ln\left\{(1-q_i) + q_i\left(1-\rho_i \ p_i^S\right)^{x_{i2}/n^S}\right\}$$

$$z_3 \geq \sum_{n=1}^{5} \gamma_n^B \ m_n^B \ C_n^B$$

$$\ln C_n^B \geq \sum_{i=1}^{2} \ln\left\{(1-q_i) + q_i\left\{\eta_n^B + (1-\eta_n^B)\left(1-\rho_i \ \sigma_{n,i}^B \ p_{n,i}^B\right)^{m_n^B x_{13}/\sum_{i=1}^{5} m_k^B}\right\}\right\}$$

$$n=1,\ldots,5$$

$$z_4 \geq \sum_{n=1}^{5} w_n^P \ \gamma_n^P \ m_n^P \ C_n^P$$

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$$\ln C_n^P \geq \sum_{i=1}^{3} \ln \left\{ (1-q_i) + q_i\left(1-\rho_i \ \sigma_{n,i}^P \ p_{n,i}^P\right)^{m_i^P \ x_{14} \Big/ \sum_{k=1}^{3} m_k^P} \right\} \qquad n=1,\ldots,3$$

All x_{ij}, z_j, ξ^R, ξ^B, C_n^B, $C_n^P \geq 0$. Note that we have changed some equalities to inequalities. This was done only to improve the efficiency of the computation. It is apparent by monotonicity that all inequalities will be equalities at the optimal solution.

The next step is to place upper and lower bounds on all the variables. This can easily be done because the allocations on each side are bounded by the available resources. The bounds appear in Table 1. In order to save space in this table, we have occasionally employed the notation $\ell(\zeta)$ and $u(\zeta)$ to denote the lower and upper bounds on variable ζ, respectively.

Because the problem comprises only scalar functions of bounded variables, and because these functions remain bounded over their domains, we can approximate these functions by piecewise-linear functions in the following way.

Each function $g(\zeta)$ in the separated problem is a function of a scalar variable for which we have lower and upper bounds $\ell(\zeta)$, $u(\zeta)$. We divide the interval $[\ell(\zeta), u(\zeta)]$ into subintervals determined by the "cut points" t^0, $t^1,\ldots,t^k$ where $t^0 = \ell(\zeta)$, $t^k = u(\zeta)$ and $t^0 < t^1 < t^2 < \ldots < t^k$. Then we compute $g(t^i)$ for $i = 0,\ldots,k$ and form the piecewise-linear function

$$g(t) = \frac{t-t^i}{t^{i+1}-t^i}\left[g(t^{i+1}) - g(t^i)\right] + g(t^i)$$

$$\text{for } t \in \left[t^i, \ t^{i+1}\right] \qquad i=0,\ldots,k-1.$$

This process is depicted in Figure 3.

The problem thus generated will be called the "piecewise-linear two-strike problem." Since all functions in the separated problem are continuous, so are all the functions of the piecewise-linear problem. Further, by taking enough cut points for each

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Table 1. LOWER AND UPPER BOUNDS

Variable	Lower Bound	Upper Bound
x_{ij} ($i=1,\ldots,4$: $j=1,\ldots,5$)	0	m_i
z_1	$w^I n^I (1-q_1)(1-q_2)$	$w^I n^I$
z_2	$n^S w^S \gamma^S \left[1-\tau^S+\tau^S(1-q_1)(1-q_2)\right]$	$n^S w^S \gamma^S$
z_3	$\sum_{n=1}^{5} \gamma_n^B m_n^B \ell(C_n^B)$	$\sum_{n=1}^{5} \gamma_n^B m_n^B$
z_4	$\sum_{n=1}^{3} w_n^P \gamma_n^P \ell(C_n^P)$	$\sum_{n=1}^{3} w_n^P \gamma_n^P m_n^P$
ξ^B	0	$\sum_{k=1}^{4} \mu_k^B m_k^{\nu_k^B}$
ξ^R	$\sum_{k=1}^{4} \mu_k^R \left[\ell(z_k)\right]^{\nu_k^R}$	$\sum_{j=1}^{4} \mu_k^R \left[u(z_k)\right]^{\nu_k^R}$
C_n^B ($n=1,\ldots,5$)	$\prod_{i=1}^{2} (1-q_i + q_i\, n_n^B)$	1
C_n^P ($n=1,\ldots,3$)	$\prod_{i=1}^{3} (1-q_i)$	1

variable, the piecewise-linear two-strike problem can be made to approximate the original two-strike problem arbitrarily closely according to any of a number of standard measures of closeness.

The piecewise-linear problem has the advantage that it can be solved using a practical branch-and-bound algorithm developed by Falk [5]. This algorithm has been programmed for the computer twice. The NUGLOBAL code [7] used by Bracken, Falk, and Miercort
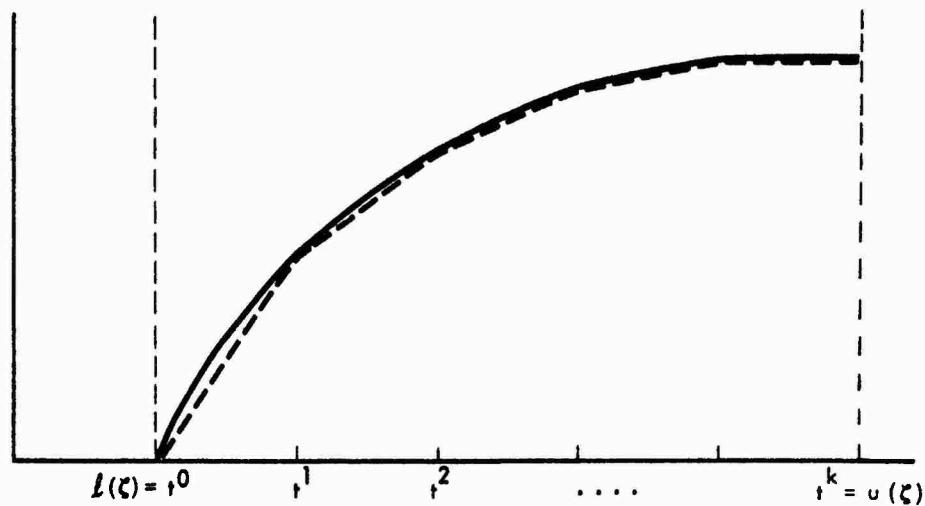
Figure 3.   PIECEWISE-LINEAR APPROXIMATION

developed problems when applied to the model we have described
above.   A more stable code was developed by Grotte [6] and was
the code used in the runs described below.

Chapter VI
RESULTS

The piecewise-linear model was run on a CDC 6400 computer.
For all runs, upper and lower bounds were computed as described
above and eight cut points per variable were chosen as follows.
Noting that all functions $g(\zeta)$ in the separated two-strike model
had derivatives that were largest in absolute value for small
ζ, and that decreased in absolute value as ζ increased, we felt
that small intervals for small ζ, and large intervals for large
ζ, should be used for constructing the piecewise-linear approxi-
mating function $g(\zeta)$. Therefore, $\ell(\zeta)$ and $u(\zeta)$ were computed
first, $r \equiv u(\zeta) - (\zeta)/ 127$ was then calculated, and the cut
points for ζ were chosen to be

$$t^i = \ell(\zeta) + (2^i - 1)r \qquad\qquad i=0,\ldots,7.$$

In this way, each subinterval was twice the length of the one
preceding it, ensuring that there would be small subintervals
at the low end of the interval $[\ell(\zeta),\ u(\zeta)]$ and larger sub-
intervals at the high end. In practice, this procedure produced
better results than using many more intervals of uniform length.

Our investigation of the responses of the model begins with
a base case set of parameters. These are listed in Table 2. A
number of series of runs were performed in each of which a specific
subset of parameters was varied in a systematic fashion. Certain
model outputs were then graphed to demonstrate the sensitivity
of the model to parameter changes. These are discussed below.
Each run took between 100 and 200 seconds of CPU time.

Table 2. BASE PARAMETERS

$V_R = 5000.0$	$m_1 = 6000.0$	$q_1 = 0.9999$	$\rho_1 = 0.8$
$V_B = 5000.0$	$m_2 = 1800.0$	$q_2 = 0.9999$	$\rho_2 = 0.8$
	$m_3 = 1500.0$	$q_3 = 0.9999$	$\rho_3 = 0.9$
	$m_4 = 1200.0$	$q_4 = 0.9900$	$\rho_4 = 0.7$
$\mu_1^B = 0.0005$	$\mu_1^R = 0.0005$	$\mu_1^B = 0.7$	$\mu_1^R = 0.7$
$\mu_2^B = 0.0003$	$\mu_2^R = 0.0003$	$\mu_2^B = 0.5$	$\mu_2^R = 0.5$
$\mu_3^B = 0.0003$	$\mu_3^R = 0.0006$	$\mu_3^B = 0.5$	$\mu_3^R = 0.5$
$\mu_4^B = 0.0003$	$\mu_4^R = 0.0003$	$\mu_4^B = 0.5$	$\mu_4^R = 0.5$
$n^I = 1000.0$	$p_1^I = 0.8$	$\sigma_1^I = 0.95$	
$w^I = 6.0$	$p_2^I = 0.7$	$\sigma_2^I = 0.95$	
$n^S = 50.3$	$p_1^S = 0.9$		
$w^S = 10.0$	$p_2^S = 0.9$		
$\gamma^S = 16.0$			
$\tau^S = 0.3$			
$N^B = 5.0$			

$\gamma_1^B = 10.0$	$m_1^B = 100.0$	$\eta_1^B = 0.5$	$\sigma_{11}^B = 0.95$	$\sigma_{12}^B = 0.95$
$\gamma_2^B = 10.0$	$m_2^B = 100.0$	$\eta_2^B = 0.5$	$\sigma_{21}^B = 0.95$	$\sigma_{22}^B = 0.95$
$\gamma_3^B = 10.0$	$m_3^B = 200.0$	$\eta_3^B = 0.4$	$\sigma_{31}^B = 0.95$	$\sigma_{33}^B = 0.95$
$\gamma_4^B = 10.0$	$m_4^B = 200.0$	$\eta_4^B = 0.4$	$\sigma_{41}^B = 0.95$	$\sigma_{44}^B = 0.95$
$\gamma_5^B = 10.0$	$m_5^B = 50.0$	$\eta_5^B = 0.6$	$\sigma_{51}^B = 0.95$	$\sigma_{55}^B = 0.95$

Table 2. BASE PARAMETERS (continued)

$$p_{11}^B = 0.97 \qquad p_{12}^B = 0.97$$

$$p_{21}^B = 0.97 \qquad p_{22}^B = 0.97$$

$$p_{31}^B = 0.97 \qquad p_{32}^B = 0.97$$

$$p_{41}^B = 0.97 \qquad p_{42}^B = 0.97$$

$$p_{51}^B = 0.97 \qquad p_{52}^B = 0.97$$

$$N^P = 3.0$$

$$w_1^P = 10.0 \quad \gamma_1^P = 16.0 \quad m_1^P = 25.0 \quad \sigma_1^P = 0.9 \quad \sigma_1^P = 0.9 \quad \sigma_1^P = 0.9$$

$$w_2^P = 10.0 \quad \gamma_2^P = 16.0 \quad {}_2^P = 25.0 \quad \sigma_2^P = 0.9 \quad \sigma_2^P = 0.9 \quad \sigma_2^P = 0.9$$

$$w_3^P = 10.0 \quad \gamma_3^P = 16.0 \quad {}_3^P = 25.0 \quad \sigma_3^P = 0.7 \quad \sigma_3^P = 0.7 \quad \sigma_3^P = 0.7$$

$$p_{11}^P = 0.85 \qquad p_{12}^P = 0.85 \qquad p_{13}^P = 0.85$$

$$p_{21}^P = 0.85 \qquad p_{22}^P = 0.85 \qquad p_{23}^P = 0.85$$

$$p_{31}^P = 0.95 \qquad p_{32}^P = 0.95 \qquad p_{33}^P = 0.95$$

A. SERIES 1--MULTIPLES OF RED'S TOTAL ARSENAL

The first series of runs demonstrates the effects of changes in the number of weapons in each Red weapon class. This was achieved by multiplying the basic values of the parameters m_1, m_2, m_3 and m_4 by 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0 so that the range of 20 percent to 100 percent of the basic Red arsenal was covered. Inspection of the resulting damage curves in Figure 4 shows that as Red's arsenal increases, Red's
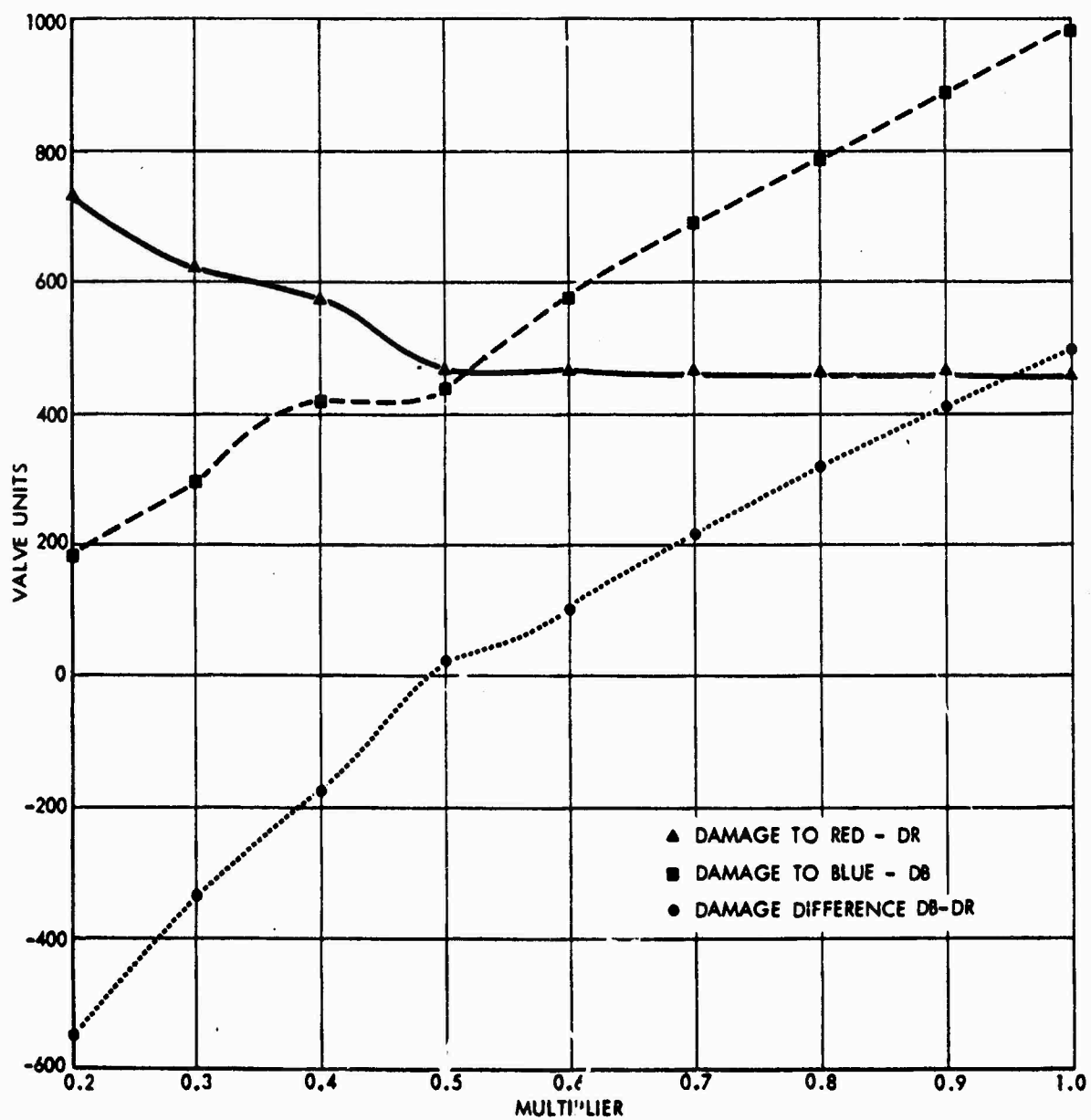
Figure 4.   VALUE DAMAGE AS A FUNCTION OF
MULTIPLIER OF RED ARSENAL

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ability to destroy Blue's value increases almost linearly, but Red's ability to protect itself from Blue retaliation does not improve for force levels above about 50 percent of the basic arsenal size. Figure 5 indicates how Red allocates weapons between counterforce and countervalue missions for various arsenal sizes. Note that for small arsenals, Red's optimal strike involves a relatively high level of counterforce operations.
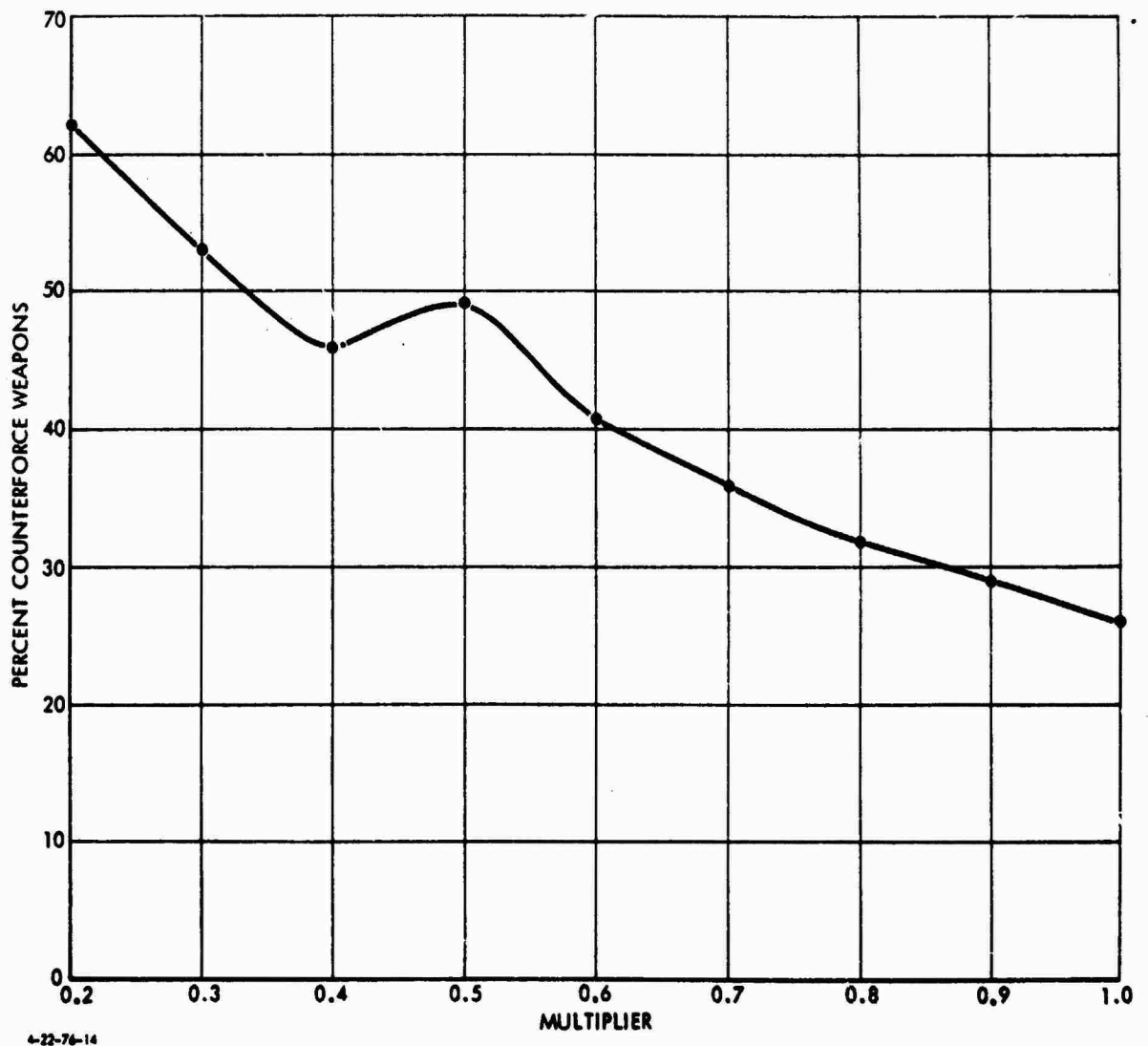

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Figure 5. PERCENT RED WEAPONS ASSIGNED TO COUNTERFORCE MISSIONS AS A FUNCTION OF MULTIPLIER OF RED ARSENAL

B. SERIES 2--CHANGES IN RED ICBM AGAINST BLUE SILO KILL PROBABILITY

For the second series of runs, we changed only p_1^I, which represents the probability of single shot kill of a Red ICBM warhead against a Blue silo. Figure 6 exhibits the resulting value damage curves. It is apparent that for values of p_1^I below about 0.5, changes in ICBM hard-target kill probability make little difference in the outcome. Observe that as p_1^I increases, the damage differences increase, which is expected; but total damage to Blue value generally *decreases*, since the more effective Red ICBMs are diverted from the countervalue attack in a successful effort to reduce retaliatory damage to Red. Note the undulatory behavior of the curves. We can interpret Figure 6 as follows: For p_1^I less than 0.4, Red's attack remains the same, resulting in level damage curves for both Blue and Red. For p_1^I between 0.4 and 0.6, Red can take advantage of increased ICBM effectiveness and so divert some of these weapons from countervalue to counterforce missions. With fewer weapons against Blue value, the damage to Blue decreases. Damage to Red decreases because of increased counterforce attack. As p_1^I increases between 0.6 and 0.7, Red finds he can accomplish the same counterforce goal with fewer ICBMs and so he reapplies the excess to countervalue assignments, thereby increasing the damage to Blue while keeping the retaliatory damage to himself constant. For p_1^I between 0.7 and 0.75, Red can cause more counterforce damage with fewer weapons so that damage to himself actually begins to drop while damage to Blue still increases. Between p_1^I = 0.75 and p_1^I = 0.9, Red again diverts weapons from value targets to Blue ICBMs, so damage to both sides decreases. For p_1^I above 0.9, the curves begin to repeat the behavior seen for p_1^I greater than 0.6. This behavior results from rapidly changing allocations as p_1^I changes, and indeed, one should note that in the case of a "flat" global optimum, or in the case of many local optima, all with values close together, the allocations
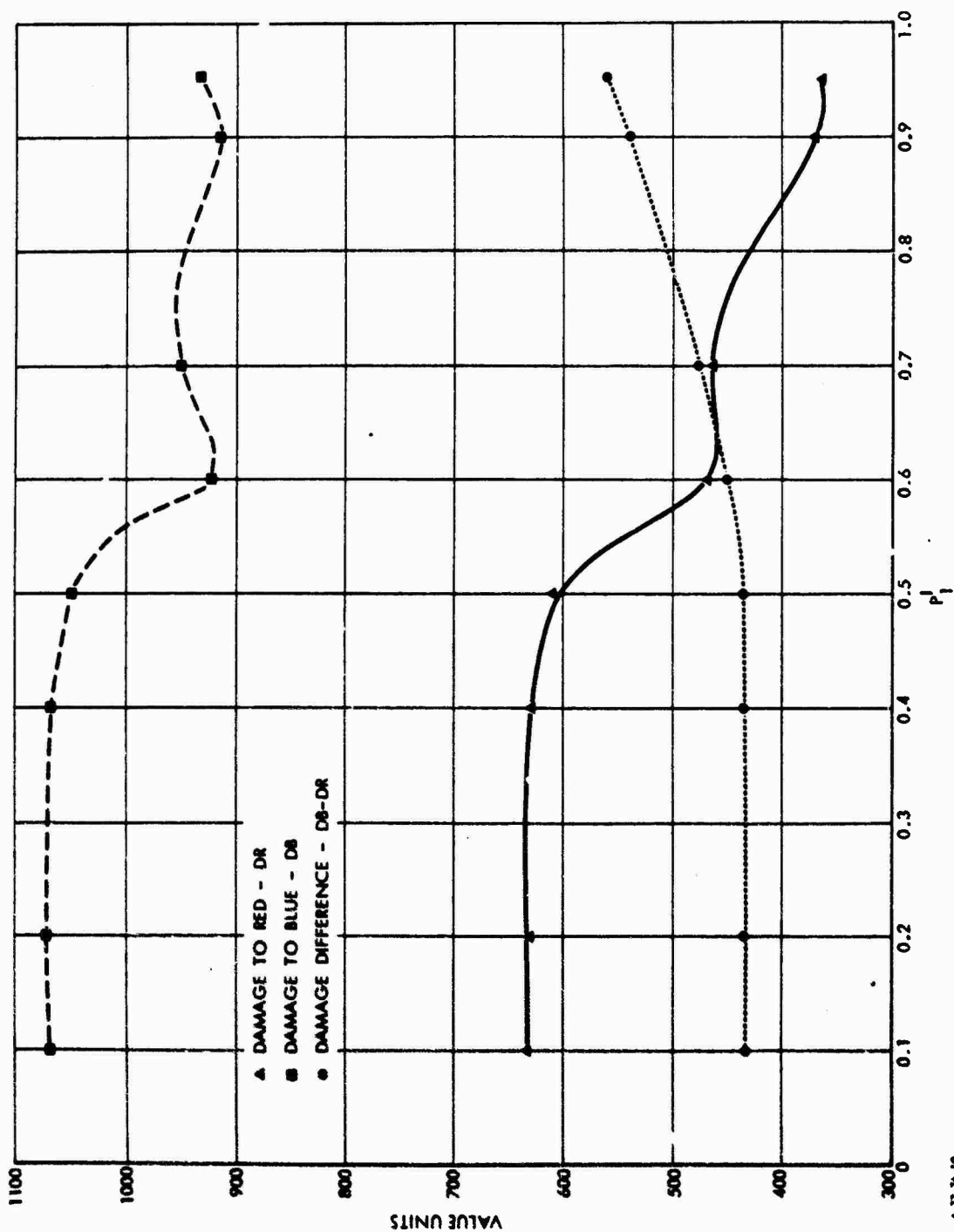
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Figure 6. VALUE DAMAGE AS A FUNCTION OF THE PROBABILITY OF A RED
ICBM KILLING A BLUE SILO, p_1^I

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may change almost discontinuously as certain parameters are varied, even though the objective function changes smoothly.

C. SERIES 3--CHANGES IN RED SYSTEM RELIABILITY

This series of runs was conducted by assigning the values 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9, one at a time, to all parameters q_1, q_2, q_3, and q_4. These parameters represent the system reliability of Red's four weapon classes against Blue weapons only. The resulting damage curves are displayed in Figure 7. Note the swift decline in the damage suffered by Red as compared with the much slower decline in the damage suffered by Blue as Red shifts weapons of increased reliability to counterforce operations. Contrast the nonmonotonic behavior of the Blue damage curve to the strictly decreasing Red damage curve and the smooth rise of the damage difference curve. Figure 8 shows how the surviving fraction of Blue warheads changes for each class. Observe that for low Red system reliabilities, Blue weapon class 1 (ICBMs) survives better than Blue weapon class 3 (Bombers), while for high Red system reliabilities, Blue weapon class 3 has the edge.

D. SERIES 4--CHANGES IN BLUE BOMBER SCRAMBLE PROBABILITIES

This series tests the effect of varying the Blue Bomber scramble probabilities. The variation was effected by assigning the values 0.5, 0.6, 0.7, 0.8, and 0.9, one at a time, jointly to n_1^B, n_2^B, n_3^B, n_4^B, and n_5^B. The damage curves appear in Figure 9. The result is that, as all the n_i^B increase Red is not induced to change his attack, but Blue can inflict greater retaliatory damage owing to his improved bomber survivability.

E. SERIES 5--MULTIPLES OF BLUE'S TOTAL ARSENAL

The last series examines what happens when Blue's arsenal is increased over the basic values, by multiplying the basic
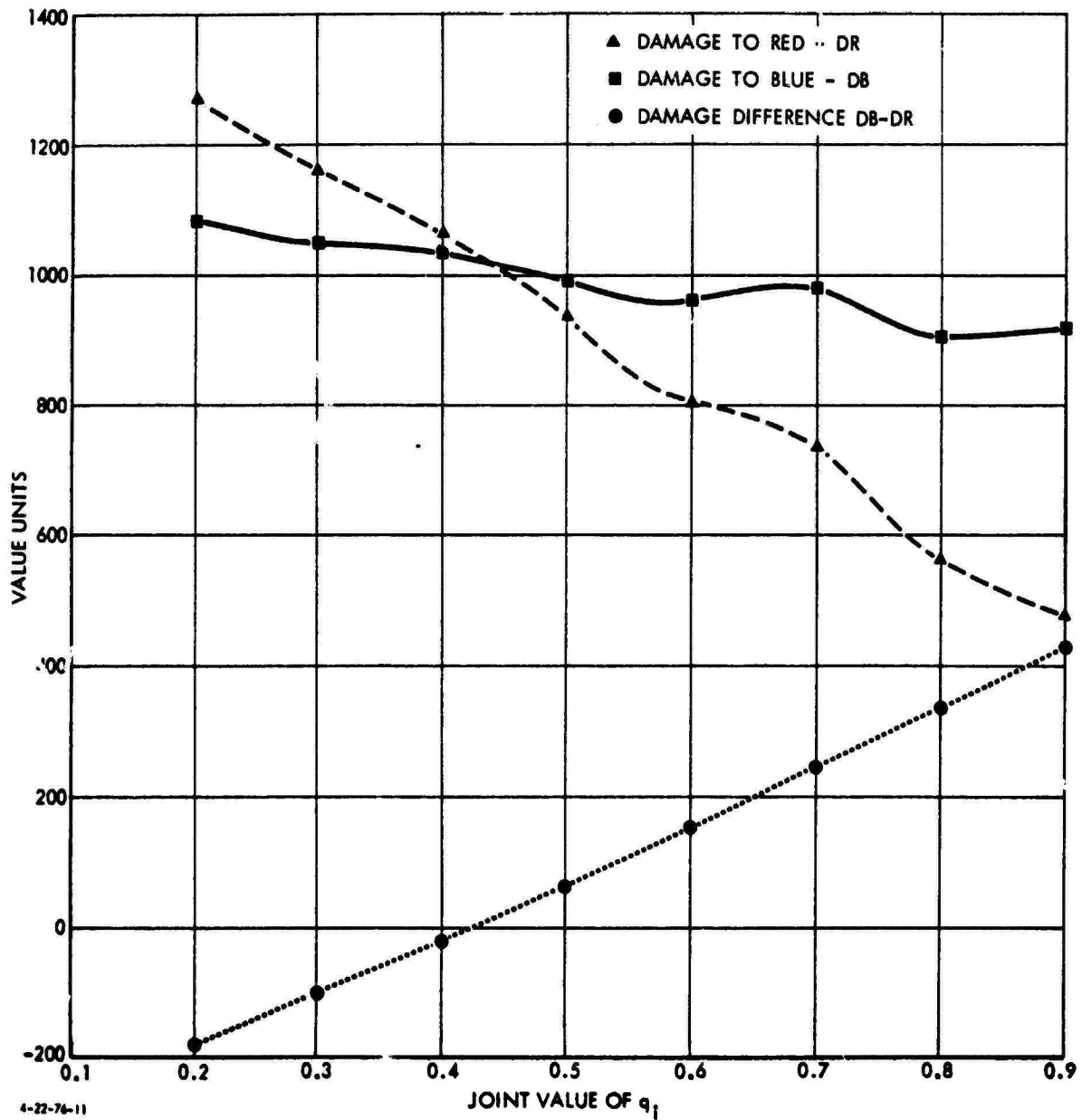
27

Figure 7. VALUE DAMAGE AS A FUNCTION OF RED SYSTEM RELIABILITY

values of all the parameters n^I, n^S, m_1^B, m_2^B, m_3^B, m_4^B, m_5^B, m_1^P, m_2^P, and m_3^P by 1.0, 1.1, 1.2, 1.3, 1.4, and 1.5. Figure 10 displays the damage curves. While an increased Blue arsenal improves Blue's retaliatory capability, it further serves a secondary role of drawing Red's attack away from Blue's value targets.
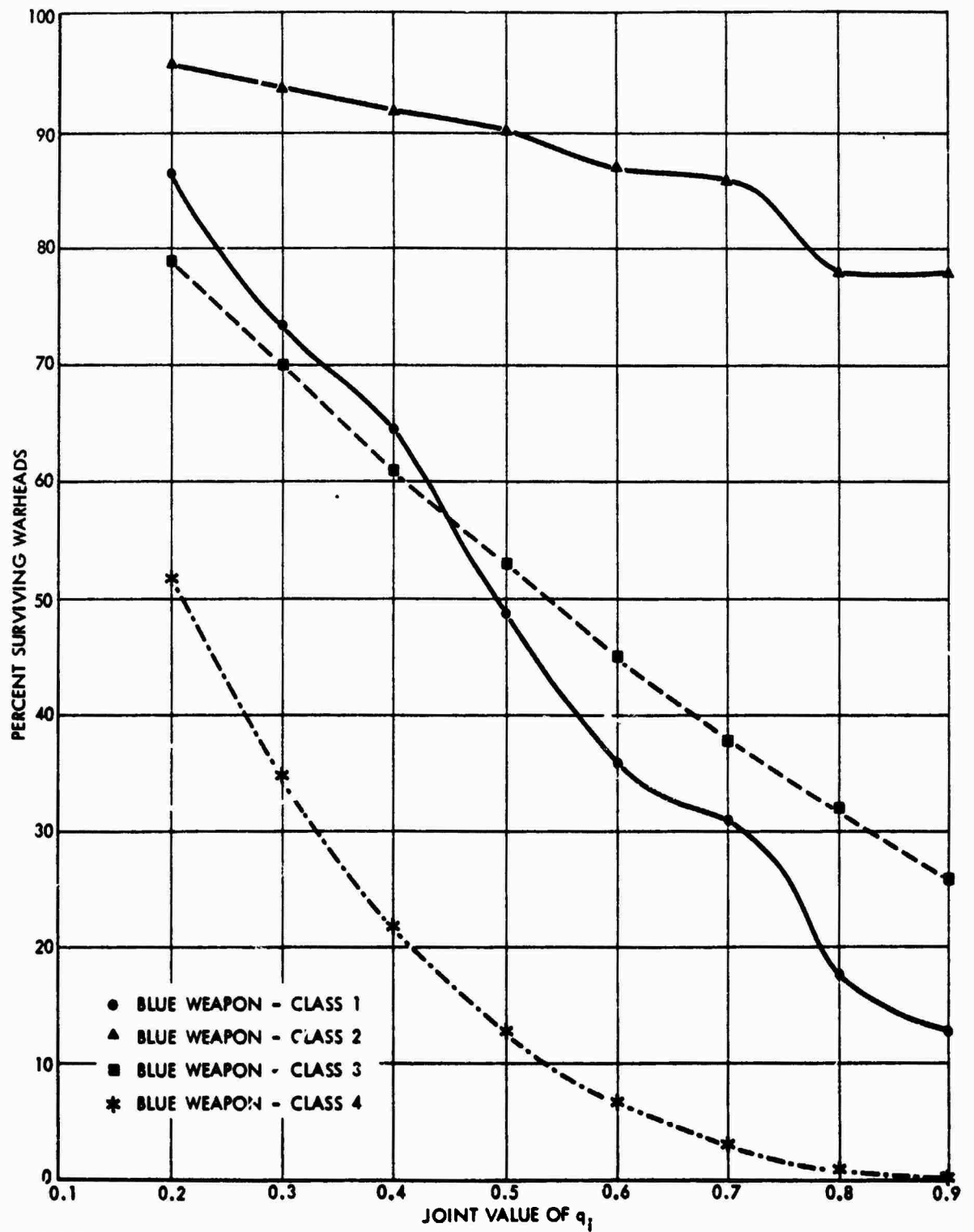
Figure 8. PERCENT SURVIVING WARHEADS AS A FUNCTION OF RED SYSTEM RELIABILITY, q_i
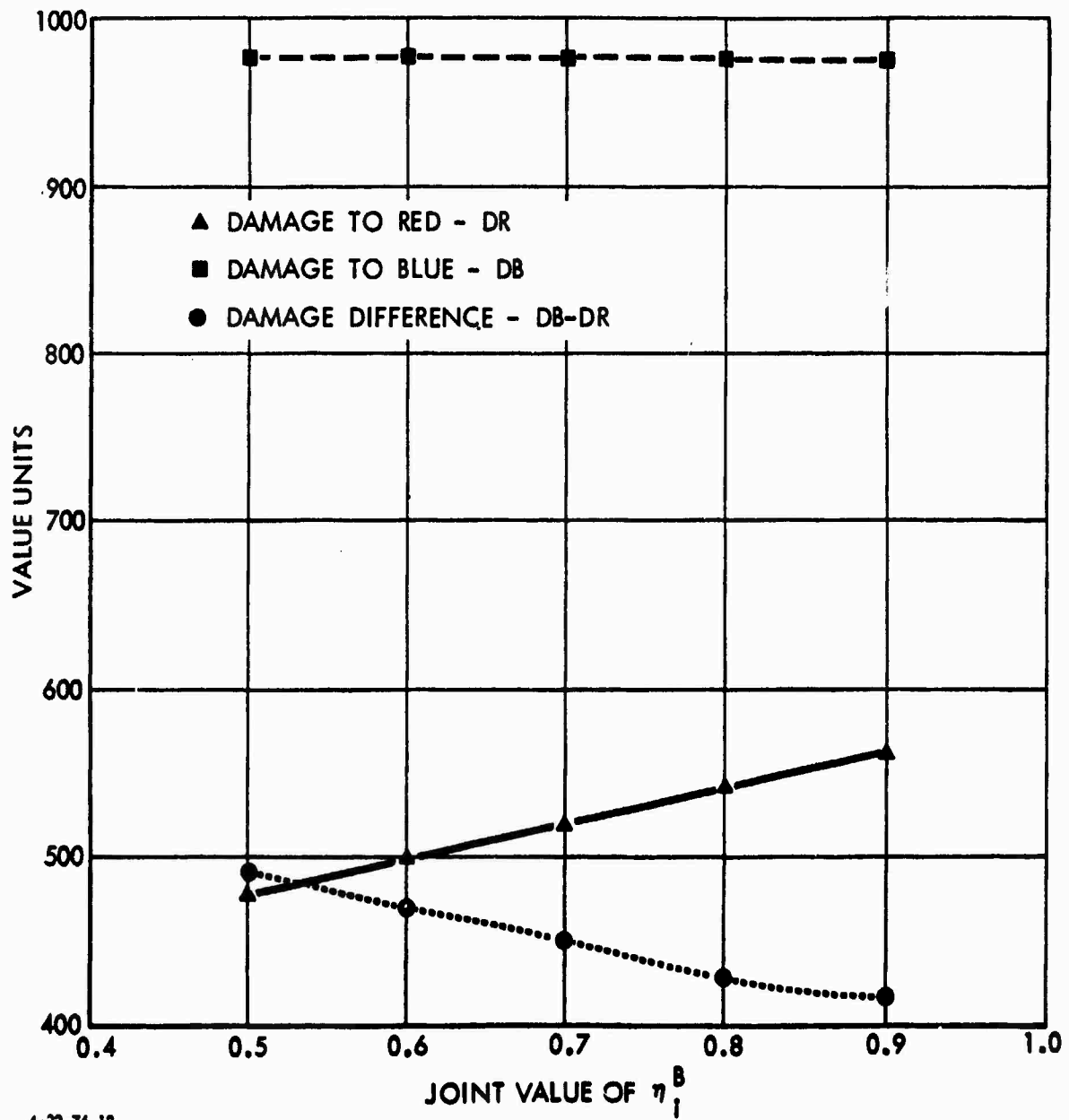
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Figure 9. VALUE DAMAGE AS A FUNCTION OF BLUE
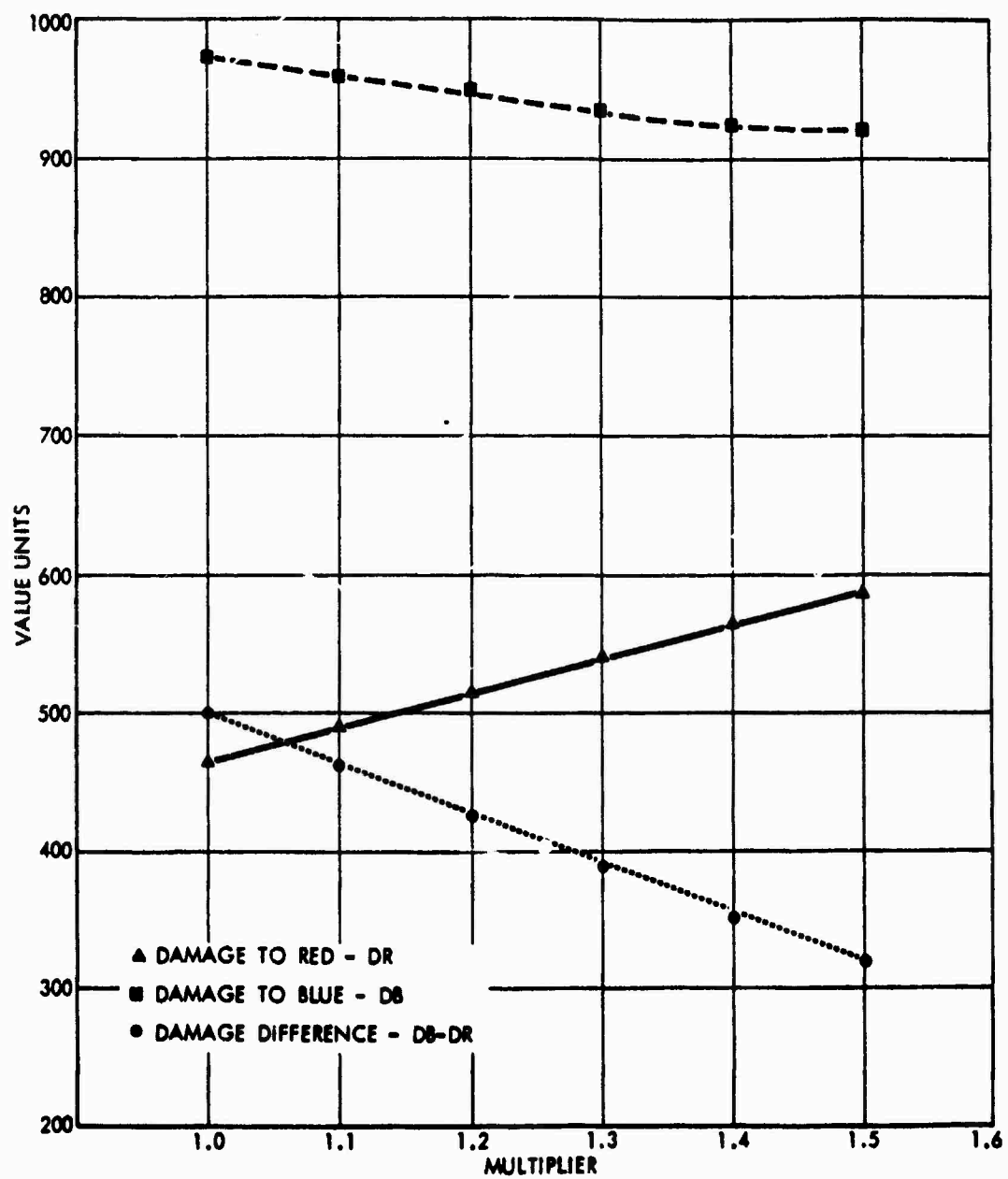BOMBER SCRAMBLE PROBABILITIES, η_i^B

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Figure 10. VALUE DAMAGE AS A FUNCTION OF MULTIPLIER
OF BLUE ARSENAL

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